PARTIALLY ORDERED SETS

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1. Preliminary review of Partially Ordered Sets, i.e. *Posets*

Article 1 titled *"Some Elementary Geometric Aspects in Extending the Dimension of the Space of Instants"* introduced the set of instants, $\mathcal{I}$, assumed to be a *poset* consisting of elements i, j, k, ... called *Instants*. As such, familiarity with the concept of a *poset* becomes essential to the theory. The following article is presented to clarify some of these concepts.

1.1. Sets and Binary Relations.

Let us begin with some brief preliminary remarks concerning *sets* and *relations*[1]. Simply speaking a set S is a collection of objects wherein its members are customarily determined by a listing of specified criteria. Any object x that satisfies these criteria can be considered *members* or *elements* of the set. When this is the case we write x $\in$ A, *i.e.* x *is contained in* A. If x fails to satisfy at least one of the criteria, then x is not an element of A and we write x $\notin$ A. Sets are usually denoted by capital letters from the beginning of the alphabet, i.e. A, B, C, ..., etc.

Examples: a) Let A = $\{x$ such that x is even $\}$ then x = 4 $\in$ A; and x = 7 $\notin$ A.
Note: Often we replace the phrase *"such that"* with a vertical bar "|" so that A may be written as: A = $\{x \mid$ x = 2n, n any integer$\}$.

b) Let A = $\{(a, b) \mid$ (a-b) is divisible by 5$\}$ then x = (-10,5) $\in$ A; and x = (4,-5) $\notin$ A.

It is sometimes necessary to consider **subsets** B of a given set A. In such a case we write B $\subseteq$ A if, and only if (iff) every member of the set B is a member of A, including A itself and say B is a **subset** of A. If only a portion of the set A is included in B, not including A, then we write B $\subset$ A and say B is a **proper subset** of A. If the set B is neither a subset nor proper subset of A we write, B $\not\subset$ A

Definition 1.1.1. The Cartesian Product of Sets

Let A and B be sets, the Cartesian Product of A and B, denoted by A x B, is *defined as* A x B = $\{$(a,b)$|$ a $\in$ A and b $\in$ B$\}$.

Note: 1. The elements (a,b) $\in$ A x B are called *ordered pairs* in A x B.

2. This definition can be extended to *finite* number of sets, i.e. A x B x C x ... etc. In the case of n ($< \infty$) repeated Cartesian products of the same set A, we write the n-copies of A as A^n = A x A x A x...A.

Examples: a) Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^3$ denote the repeated

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[1]For a more thorough treatment of sets and relations, see *Topics in Algebra*, I. N. Herstein

Cartesian product of $\mathbb{R}$, i.e. $\mathbb{R}^3 = \mathbb{R}$ x $\mathbb{R}$ x $\mathbb{R}$.

b) Geometrically if $\mathbb{R}^3$ corresponds to three mutually perpendicular real number lines then $\mathbb{R}^3$ is the usual rectangular Cartesian system.

Definition 1.1.2. Binary Relation

A binary relation R is a subset of A x B, *that establishes a rule or relation* $\mathcal{R}$ *between elements within the ordered pairs of R.*

Note: 1. If B = A, and R is a subset of A x A, then R is said to be a ***binary relation on A.***
2. Note the distinction between the set R and the relation rule $\mathcal{R}$.

Examples: a) Let S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and consider the set R $\subseteq$ S x S such that the (binary) rule $\mathcal{R}$ for (a,b) $\in$ R is defined by (a $\mathcal{R}$ b) if b = 2a. Then R = {(1,2), (2,4), (3,6), (4,8), (5,10)} $\subset$ (S x S).

b) Let S = {Charles, Jane, Allison, Robert, 0, 1, 2, 3} and let the set R $\subseteq$ S x S such that the (binary) rule $\mathcal{R}$ for (a,b) $\in$ R is defined as " named person owns 0, 1, 2, or 3 cars". Then R = {(Charles, 2), (Allison, 3), (Robert, 0)} $\subset$ (S x S), but (0, 1) and (Robert, Allison) are not in the relation defined by $\mathcal{R}$.

Note: 1. It is customary to refer to refer to binary relations R on a set S without regard to the rule $\mathcal{R}$ where the underlying rule for the relation is assumed known.
2. One often uses $\sim$ or $\bowtie$ as alternate symbols for the rule $\mathcal{R}$.

Definition 1.1.3. Reflexive, Symmetric, Transitive, Irreflexive, and Antisymmetric Binary Relations.

If R is a binary relation on a set S, then
i) R is ***reflexive*** if and only if (x, x) $\in$ R, $\forall$ x $\in$ S.
ii) R is ***symmetric*** if and only if, for any x and y in S, whenever (x, y) $\in$ R then (y, x) $\in$ R.
iii) R is ***transitive*** if and only if, for any x, y, and z in S, if (x, y) $\in$ R and (y, z) $\in$ R then (x, z) $\in$ R.
iv) R is ***irreflexive*** if and only if (x, x) $\notin$ R for all x $\in$ S.
v) R is ***antisymmetric*** if and only if, if (x, y) $\in$ R and (y, x) $\in$ R then y = x, for all x and y $\in$ S.

Examples: a) Let the set S be the set of integers with R being the usual *less than* i.e., "<" on S. Then R is neither reflexive (since (x, x) is not in R) nor symmetric, but clearly it is transitive since x < y and y < z does imply that x < z.
b) For R defined as the usual $\leq$ over the reals, then R is clearly reflexive but not symmetric.

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1.2. **Partially Ordered Sets.**

Often elements of a set may be compared in terms of an *ordering*. These orderings come in several varieties, *viz.,* total orderings, well orderings, semi-orderings, partial orderings, etc. For our purposes we will consider *total* and *partial* orderings. Simply speaking a total ordering on a set S is defined by a relation such that $\forall$ x, y in S, x and y are *comparable* in the sense that exactly one of three possibilities holds: x is related to y; y is related to x; or x =y. In contrast, when this *triconomy* holds for only some elements of the set then the set is called *partially ordered*. To be exact we define a partially ordered set.

Definition 1.2.1. Partially Ordered Set

A partially *ordered set (or poset) is a set S with a binary relation R satisfying the conditions*:

i) **reflexivity**, i.e. (x, x) ∈ R, ∀ x ∈ S.
ii) **antisymmetry**, i.e.(x, y) ∈ R and (y, x) ∈ R then y = x, for all x and y ∈ S.
iii) **transitivity**, i.e., if (x, y) ∈ R and (y, z) ∈ R then (x, z) ∈ R.

Examples: a) Let A ={1, 2} and consider the be the set of subsets of A then ⊆ is a partial order on the set of subsets S = {∅, {1}, {2}, {1,2}}.

b) Let A = {1,2,3,4,6,12} with the binary relation R = x|y for (x,y) ∈ A x A thus {1,2}, {1,4}, {1,6}, {1,12}, ... etc. is a partial ordering.

These are just some of the simplest examples of partial sets and, as aforementioned is not inte
recommended sources:
a) Abstract Algebra by I. N. Herstein (there are several editions each is very good)
b) Topics in Algebra, 2nd Edition Jun 20, 1975 by I. N. Herstein
c) Algebra by Serge Lang (this is more advanced but an excellent text)
d) Basic Algebra I: Second Edition by Nathan Jacobson