SOME ELEMENTARY GEOMETRIC ASPECTS IN EXTENDING THE DIMENSION OF THE SPACE OF INSTANTS

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Abstract

A local geometric construction is proposed on the *partially ordered* set of instants $\mathcal{I}$. A totally ordered subset $\mathcal{C}(\mathcal{I}) \subset \mathcal{I}$ is assumed to have 3-dimensional affine coordinate structure, *without* a specified metric, called the τ-space of $\mathcal{C}(\mathcal{I})$. Guided by a strong analogy with analytical mechanics the *T-configuration space* (θ, τ^{α}), θ a real parameter, is constructed whereupon the usual Hamilton-Jacobi theory establishes a simple geometrical construction, *viz.*, the complete figure from the calculus of variations. The duration function, dur:$\mathcal{C}(\mathcal{I}) \to \mathbb{R}$, is associated with *temporally equidistant* hypersurfaces through which pass a congruence of extremal curves to the fundamental integral.

1. Introduction

In a recent paper by DeVito (DeVito, 1995), it has been suggested that time, because of its association with the real line, has escaped a more general local topological inquiry. As such, the usual distinction between space and time is evident with space being endowed with a rich geometric and topologic structure. However, when considering the configuration space of a particle in a dynamical setting, *viz.*, (t, x^i), $i = 1, ..., n$, time is treated simply as an additional coordinate of space. In this respect, both space and time have been regarded as geometrical objects of the same kind. As such, a likely extension is to consider time as a multi-dimensional geometric quantity as one does naturally when considering the structure of space alone. This is neither a new idea nor one ignored, since multiple dimensions of time have been discussed previously by Cole (1980), Cole and Starr (1990), Lehto (1990), and Tifft (1995). Altough some interesting physical consequences have arisen in particle decay, relativity, and cosmology, this approach has not been problem free. Certainly more questions have been raised than answered as to the physical meaning of extending the dimension of time. Cole in 1979 introduced a 3-d modification to the temporal component in the usual metric used in general relativity for the static spherically symmetric case. The resulting solution predicted an unsatisfactory Mercury perihelion advance 7/3 times that predicted in usual 4-d GR (Cole 1980).

In an attempt to address these problems it has been suggested (DeVito 1995) that a mathematical model of time be developed, *ab initio*, without a component of space, while still preserving the usual properties attributed to our familiar concept of, what we now refer to as, *observable time*. To this end Devito has introduced a *partially ordered set* $\mathcal{I}$ consisting of elements $i, j, k, ...$ called *instants* with no immediate or apparent connection to our observable time.

Together with the partial ordering $\leq$ on $\mathcal{I}$, we say that i and j are *comparable* if $i \leq j$ or $j \leq i$ and we let

$$\mathcal{C}(\mathcal{I}) = \{(i,j) \in \mathcal{I} \times \mathcal{I} \mid i \text{ and } j \text{ are comparable}\}.$$

Also, defined on $\mathcal{C}(\mathcal{I})$ is a map to the non-negative reals $\mathbb{R}_+$ called the "duration" function denoted by $\mathrm{dur}(i,j)$ such that:

$$\text{a) } \mathrm{dur}(i,j) = 0 \text{ iff } i = j; \tag{1}$$
$$\text{b) } \mathrm{dur}(i,j) = \mathrm{dur}(j,i) \text{ for any } (i,j) \text{ in } \mathcal{C}(\mathcal{I}).$$

Remark. We note that the duration function, being defined for all comparable i, j in $\mathcal{I}$, is globally defined and that no distinction is made between (i,j) and (j,i).

Essential to this theory is the definition of a *time track* on $\mathcal{I}$:

Definition 1. A non-empty set $\mathcal{C} \subseteq \mathcal{I}$ is a *time track* on $\mathcal{I}$ if:

a) $\leq$ is a total ordering on $\mathcal{C}$;

b) If $i, j, k \in \mathcal{I}$ with j between i and k, then $\mathrm{dur}(i,k) = \mathrm{dur}(i,j) + \mathrm{dur}(j,k)$;

c) For any fixed $i \in \mathcal{I}$ and any fixed $\rho \in \mathbb{R}_+$ there are exactly two *distinct* $x, y \in \mathcal{C}$ such that $\mathrm{dur}(i,x) = \rho = \mathrm{dur}(i,y)$.

In this paper we adhere closely to this structure; however, in addition to some notational modifications we do impose a significant additional assumption on $\mathcal{I}$, namely, that the set $\mathcal{I}$ may be considered within the context of an n-dimensional space $\mathbb{R}^n$. However, at this early stage it must be emphasized that a metric on this space has not been specified nor should one be presumed.

Earlier attempts at extending such a coordinate structure to time while preserving the standard Euclidean metric have met with considerable difficulty. As mentioned earlier, in the context of general relativity, Cole (1980) introduced a metric for the static spherically symmetric case that included three dimensions of time which was given as

$$ds^2 = -a(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\psi^2) + b(r)[(dt^1)^2 + (dt^2)^2 + (dt^3)^2].$$

This, however, resulted in predicting an orbital perihelion shift, for the case of Mercury, over twice that predicted using 4-d GR. This approach immediately suggests two problems: firstly, the incorporation of such a temporal decomposition into the usual space-time manifold structure; and secondly, the assumption that *observable time* can be expressed with the usual Euclidean metric, i.e., $dt^2 = (dt^1)^2 + (dt^2)^2 + (dt^3)^2$.

Thus, our initial assumption, *viz.* to consider $\mathcal{I}$ apart from space, is a necessary one, and as such $\mathcal{I}$ does not immediately inherit those properties one normally attributes to space in various physical settings. In this regard we introduce the next section.

2. Mathematical Foundations

I. Preliminaries. We begin our discussion with the following assumptions and definitions. Let the sets $\mathcal{I}$ and $\mathcal{C}(\mathcal{I})$ be given and defined as above. These sets are said to consist, respectively, of *points* called *instants* $P, Q, R, ... \in \mathcal{I}$ and *comparable points* $(P,Q) \in \mathcal{C}(\mathcal{I})$ with $\mathrm{dur}:\mathcal{C}(\mathcal{I}) \to \mathbb{R}_+$ as defined by (1). We note that, by virtue of equation (1a,b) and definition 1, the duration function does not satisfy the triangle inequality for points R between P and Q and, as such, is not a metric on the set $\mathcal{I}$. $\mathcal{I}$, therefore, is not *metrizable* by $\mathrm{dur}(P,Q)$ and is not endowed with the *usual* topology. Because of this and the desire to keep to a minimum the number of assumptions imposed on the structure of $\mathcal{I}$, we are hesitant, at this early stage, to introduce any other topologies on $\mathcal{I}$. However, it is immediately recognized that such considerations are necessary, in the sequel, if a *meaningful* theory is to be unearthed here, particularly as it relates to physics. Of course the desired approach is to map open sets of $\mathcal{I}$ homeomorphically to open sets of $\mathbb{R}^n$. However, at this early stage we do not assume such a structure, and we note that by virtue of definition 1, a *time track* has the structure of the real line $\mathbb{R}$, and prefer to begin our discussion with a *local theory* based on the following assumptions:

1. A point P in the set of instants may be considered a point in *observable time* with no connection to *observable space.*
2. The set $\mathcal{C}(\mathcal{I})$ has a local three dimensional affine coordinate structure called the *τ-space* which we denote by τ.
3. There exists a 1-1 map by which $P \in \mathcal{C}(\mathcal{I})$ is mapped to $\tau, \phi : \mathcal{C}(\mathcal{I}) \to \tau$:

$$\phi(P) = (\tau^\alpha), \alpha, \beta, \ldots = 1, 2, 3. \tag{2}$$

Definition 2. Any set which consists of the above elements is said to have a *τ-Space Structure.*

To introduce a geometrical picture to the *τ-Space Structure* we introduce a parameter, $\phi \in \mathbb{R}$, and consider the $(\alpha+1)$-dimensional space $\mathbb{R}^{\alpha+1}$ called the *T-configuration space* denoted as (θ, τ^α).

Remark. No attempt is made here to disguise this as being anything other than analogous to the usual construction in classical analytical mechanics wherein (t, x^α) is considered the configuration space of a particle through which the particle's motion is described. In fact, we are strongly guided by such an analogy in our attempts at discovering the physics, if any, of extending the dimension of $\mathcal{I}$. However, we must be reminded that although the construction is similar, there is no space component to the *T-configuration space* and, more importantly, no metric is specified on $\mathbb{R}^{\alpha+1}$ at this early stage.

Guided by this mechanical analogue, we wish to consider a geometric model based upon the Hamilton-Jacobi theory, wherein, consistent with the above construction, a 'complete figure' (Rund 1973) in the *T-configuration space* is introduced. To this end, we require the introduction of a function on $\mathbb{R}^{\alpha+1}$ which is analagous to a Lagrangian, from which is constructed a Hamiltonian in the usual way; however, it must be emphasized that an *ad hoc* approach in forming such a construction, by analogy alone, is immediately suspect in establishing a meaningful theory. Thus, we introduce a general theorem (see Appendix) (Osgood 1946).

Theorem 1. Let $f(\xi^i),\ i, j, k, \ldots = 1, \ldots, n$ be an arbitrary C^2 function on $\mathbb{R}^n$ such that

$$\partial(f_{,1}, \ldots, f_{,n})/\partial(\xi^1, \ldots, \xi^n) \neq 0 \tag{3}$$

$$\text{where } f_{,j} = \partial f/\partial \xi^j. \tag{4}$$

Let T denote the transformation

$$y_j = f_{,j} \qquad j = 1, \ldots, n. \tag{5}$$

Now, assuming the summation convention holds, if the function $h(y^j)$ is constructed as

$$h(y_j) = \xi^k y_k - f(\xi^j) \tag{6}$$

where $\xi^j = \xi^j(y_k)$ is determined from T^{-1} (by virtue of (3)), then T^{-1} is given by:

$$\xi^j = \partial h/\partial y_j = h_{,j} \tag{7}$$

$$\text{with } \partial(h_1, \ldots, h_n)/\partial(y_1, \ldots, y_n) \neq 0. \tag{8}$$

To summarize we write (6) as:

$$f(\xi^j) + h(y_j) = \xi^k y_k \tag{9}$$

where ξ^j and y_j are given, respectively, by (5) and (7) with their respective Hessian determinants non-zero. As will be immediately evident, a useful corollary (see Appendix) now follows:

Cor. If f and h are defined as in the theorem, and, if each depends on a parameter λ, then

$$\partial f/\partial\lambda + \partial h/\partial\lambda = 0. \tag{10}$$

Remarks: 1. It is immediate from the proof of the corollary that (10) holds in the case of any finite number of parameters, λ^k.

2. The above construction is a general result and, as such, is not necessarily linked to a mechanical system; however, if $f(\lambda^k, \xi^j)$ is associated with a given *Lagrangian* $L(t, x^j, x'^j)$, defined on $\mathbb{R}^{2n+1}$ where $x^j = x^j(t)$, with $x'^j \equiv dx^j/dt = \xi^j$ with t an x^j regarded as parameters, then theorem 1 and its corollary prove Hamilton's Equations when the curves x^j satisfy the Euler-Lagrange equation $E_j(L) = 0$, $d/dt\{\partial L/\partial x'^j\} = \partial L/\partial x^j$.

For, suppose we set $y_j \equiv p_j = \partial L/\partial x'$, as in (5), and construct the function $h(y_j)$ according to the prescription given by (6), we then obtain, noting condition (3),

$$h(y_j) \equiv H(t, x^j, p_j) = -L(t, x^j, x'^j(p_k)) + p_j x'^j(p_k).$$

The desired result now follows, by using (7) with $\xi_j = x'^j$, followed by (10), with $E_j(L) = 0$:

$$x'^j = dx^j/dt = \partial H/\partial p_j,$$

$$dp_j/dt = \partial L/\partial x^j = -\partial H/\partial x^j.$$

In a similar fashion theorem 1 and its corollary may be used to prove the converse.

3. We note that the underlying theory lies solely within the context of the Calculus of Variations and that the functions $x^j(t)$ are thus regarded as defining a curve γ connecting two *fixed* points p and q in the configuration space, $\mathbb{R}^{n+1}$, of a particle in the variables (t, x^j, x'^j):

$$\gamma : x^j = x^j(t), j \qquad j = 1, \ldots, n. \tag{11}$$

Also, the n functions $x^j = x^j(t)$, possessing derivatives,

$$x'^j = dx^j/dt \tag{12}$$

satisfying $E_j(L) = 0$ are such that, for a given C^2 Lagrangian L, the fundamental integral I, taken between p and q,

$$I = \int_p^q L(t, x^j, x'^j)dt \tag{13}$$

assumes an extreme value as compared with other functions $\bar{x}^j = \bar{x}^j(t)$ which coincide at the endpoints.

II. The T-Configuration Space. The point of view we wish to adopt here is one motivated by the above remarks, and, as such, we wish to regard the *T-configuration space*, (θ, τ^α), (or T-Space) in much the same way. To this end, we shall regard θ as a single parameter and write $\tau^\alpha = \tau^\alpha(\theta)$ as defining a curve γ:

$$\gamma : \tau^\alpha = \tau^\alpha(\theta) \tag{14}$$

in the $\mathbb{R}^{\alpha+1}$ *T-configuration space* of the variables (θ, τ^α).

It is recognized that θ in this setting is simply being regarded in the same manner in which "time", t, has been regarded previously. However, with no physical significance assigned to θ, there is a fundamental difference. It is introduced here, in this parametrical setting, in the desire to extract the physical significance (if any) of multi-dimensional time. Thus, let the two *fixed* points

p, q with respective coordinates $(\theta_1, \tau_1^\alpha)$, $(\theta_2, \tau_2^\alpha)$ be the respective *fixed* endpoints on γ. Also, the quantities $\tau'^\alpha \equiv d\tau^\alpha/d\theta$ are interpreted as the components of the *tangent vector* $(1, \tau'^\alpha)$ of γ. Let us now suppose that we are given a C^2 function, $\mathcal{L}(\theta, \tau^\alpha, \tau'^\alpha)$, in the $2\alpha + 1$ arguments $(\theta, \tau^\alpha, \tau'^\alpha)$, which we identify with the function $f(\xi^j)$ in theorem 1, with θ and τ^α regarded as parameters.

We further assume that it satisfies condition (3), *viz.*,

$$\partial(\mathcal{L}_{,1}, \dots, \mathcal{L}_{,n})/\partial(\tau'^1, \dots, \tau'^n) \equiv \det(\partial^2\mathcal{L}/\partial\tau'^i\partial\tau'^j) \neq 0. \tag{15}$$

Thus, in accordance with the prescription of the theorem, with $\xi^\alpha \equiv \tau'^\alpha$ and $y_\alpha \equiv p_\alpha$, we define the transformation T as:

$$p_\alpha = \partial\mathcal{L}/\partial\tau'^\alpha \tag{16}$$

so that T^{-1}, by (7), becomes

$$\tau'^\alpha = \partial\mathcal{H}/\partial p_\alpha \tag{17}$$

where, because of (6), $h(y_j) \equiv \mathcal{H}(p_\alpha) = \tau^\beta p_\beta - \mathcal{L}(\tau'^\alpha)$, which we express as

$$\mathcal{L}(\tau'^\alpha) + \mathcal{H}(p_\alpha) = \tau^\beta p_\beta. \tag{18}$$

Remarks: 1. By virtue of (15), we can solve for the τ'^j as functions of $(\theta, \tau^\alpha, p_\alpha)$, and write

$$\tau'^\alpha = \varphi(\theta, \tau^\alpha, p_\alpha). \tag{19}$$

Thus, we now express $\mathcal{H}$ in all of its arguments

$$\mathcal{H}(\theta, \tau^\alpha, p_\alpha) = \tau^\beta p_\beta - \mathcal{L}(\theta, \tau^\alpha, \varphi(\theta, \tau^\alpha, p_\alpha)), \tag{20}$$

and by (8) we have

$$\partial(\mathcal{H}_1, \dots, \mathcal{H}_n)/\partial(p_1, \dots, p_n) = \det(\partial^2\mathcal{H}/\partial p_\alpha\partial p_\beta) \neq 0. \tag{21}$$

2. In our continued attempt to maintain a strong analogy with analytical mechanics we call $\mathcal{L}(\theta, \tau^\alpha, \tau'^\alpha)$ the *T-Space Lagrangian*, and $\mathcal{H}(\theta, \tau^\alpha, p_\alpha)$ the associated *T-Space Hamiltonian*, written in terms of p_α, which we call the *generalized T-Space momentum*. (These definitions are for identification purposes only and obviously no physical meaning is implied nor should be inferred here.) Also, we denote the *Euler-Lagrange* equation on T-Space as

$$\mathcal{E}_\alpha(\mathcal{L}) \equiv d/d\theta\{\partial\mathcal{L}/\partial\tau'\} - \partial\mathcal{L}/\partial\tau \tag{22}$$

so that $\mathcal{E}_\alpha(\mathcal{L}) = 0$ constitutes the necessary condition to be satisfied by the curves $\tau^\alpha = \tau^\alpha(\theta)$ in order that the fundamental integral

$$I = \int_p^q \mathcal{L}(\theta, \tau^j, \tau'^j) d\theta \tag{23}$$

be extremalized.

3. The function $\mathcal{H}$, in terms of $(\theta, \tau^\alpha, p_\alpha)$, is thus said to be derived from the function $\mathcal{L}$, in terms of $(\theta, \tau^\alpha, \tau'^\alpha)$, by virtue of a *T-Legendre transformation* of these variables.

We may summarize the results the results of theorem 1, *viz.*, equations (7) and (10), taken with (19), in the form of the following lemmas:

Lemma 1. The T-Hamiltonian function is of class C^2 and satisfies the identities:

$$\tau'^\alpha = \varphi(\theta, \tau^\beta, p_\beta) = \partial\mathcal{H}(\theta, \tau^\beta, p_\beta)/\partial p_\alpha \tag{24}$$

$$\partial\mathcal{H}/\partial\tau^\alpha = -\partial\mathcal{L}/\partial\tau^\alpha \tag{25}$$

$$\partial\mathcal{H}/\partial\theta^\alpha = -\partial\mathcal{L}/\partial\theta^\alpha \tag{26}$$

Lemma 2. If $\gamma : \tau^\alpha = \tau^\alpha(\theta)$ in T-Space satisfies $\mathcal{E}_\alpha(\mathcal{L}) = 0$ then

$$d\tau^\alpha/d\theta = \partial\mathcal{H}/\partial p_\alpha \tag{27}$$

$$dp_\alpha/d\theta = -\partial\mathcal{H}/\partial\tau^\alpha \tag{28}$$

III. The Hamilton-Jacobi Equation on T-Space. Let us next consider an *arbitrary* one-parameter family of hypersurfaces on the T-Space of the variables (θ, τ^α) in terms of the parameter χ characterized by the C^2 function $S(\theta, \tau^\alpha)$,

$$S(\theta, \tau^\alpha) = \chi. \tag{29}$$

We assume this family to cover a region $\mathcal{R}$ of T simply, thus to each point $\mathbf{p}$ in γ there is a unique χ belonging to the family. Let γ be a C^2 curve in T

$$\tau^\alpha = \tau^\alpha(\theta) \tag{30}$$

which intersects the family (29), nowhere tangentially to any one of its members, and is such that the components of tangent vector τ'^α are so as to generate a direction to *minimize* the fundamental integral (23). We call this direction the *temporal gradient* which is given, in general, by the condition (Rund 1976, pg 20)

$$\partial\mathcal{L}/\partial\tau'^\alpha = (\mathcal{L}/\chi')\partial S/\partial\tau^\alpha \tag{31}$$

where $\chi' = d\chi/d\theta$.

The *temporal gradient* direction together with (31) and the definition of the T-momentum (16) immediately suggest that a condition be placed on the selection of hypersurfaces, (29). Let the family of hypersurfaces which satisfy the condition

$$\mathcal{L} = \chi' \tag{32}$$

be referred to as being *temporally equidistant with respect to the T-Lagrangian* $\mathcal{L}(\theta, \tau^\alpha, \tau'^\alpha)$. So that under this condition we have

$$p_\alpha = \partial\mathcal{L}/\partial\tau'^\alpha = \partial S/\partial\tau^\alpha, \tag{33}$$

from which we have by (19), $\tau'^\alpha = \varphi(\theta, \tau^\alpha, p_\alpha)$. Thus by virtue of (33) and (29) we obtain a set of first order ODEs

$$\tau'^\alpha = F^\alpha(\theta, \tau^\beta), \tag{34}$$

the solutions of which yield a 3-parameter family of curves called the T-congruence of curves belonging to the family of hypersurfaces $S(\theta, \tau^\alpha) = \chi$. These results may now be used to establish the next lemma (Rund 1976):

Lemma 3. Given $\mathbf{p}$ an arbitrary point on a given hypersurface $S(\theta, \tau) = \chi_1$ with $\mathbf{q}$ a member of the *T-congruence*, belonging to the family, which intersects any other member of the family with parameter valu χ_2, then

$$I = \int_p^q \mathcal{L}(\theta, \tau^\alpha, \tau'^\alpha)d\theta = \chi_1 - \chi_2 \tag{35}$$

and is independent of the position of the *initial point* $\mathbf{p}$ on the first hypersurface.

Remarks: 1. In a sense, we may say that two temporally equidistant hypersurfaces, characterized by their respective parameters χ_1 and χ_2, cut off "equal parts" from every member of the congruence belonging to the family.

2. We are thus motivated to establish a relationship between such hypersurfaces and the *duration function*, $\mathrm{dur}(P, Q)$, on the Set of Instants $\mathcal{I}$, wherein, for I given by (35):

$$\mathrm{dur}(P, Q) \simeq I. \tag{36}$$

3. Before establishing such a relationship we must specify a kind of *reality condition* that we impose on the nature of the curves characterized by (30) that is consistent with "observable time." To this end we recognize that to each point $\mathbf{p}$ in γ there is associated a unique χ which generates the hypersurface (29), and as such, χ may be regarded as a monotonic function of θ along γ to which we impose this *reality condition*, *viz.*,

$$\chi' = d\chi/d\theta > 0. \tag{37}$$

We state the following theorem (Rund 1976) which establishes the necessary and sufficient conditions which must be satisfied by the family of hypersurfaces in order that they be *temporally equidistant with respect to the T-Lagrangian* $\mathcal{L}(\theta, \tau^\alpha, \tau'^\alpha)$.

Theorem 2. A family of hypersurfaces $S(\theta, \tau^\alpha) = \chi$ is *temporally equidistant* with respect to the *T-Lagrangian* $\mathcal{L}(\theta, \tau^\alpha, \tau'^\alpha)$ if, and only if, $S(\theta, \tau^\alpha)$ is a solution of the Hamilton-Jacobi equation,

$$\partial S/\partial\theta + \mathcal{H}(\theta, \tau^\alpha, \partial S/\partial\tau^\alpha) = 0 \tag{38}$$

$$\text{with } p_\alpha = \partial S/\partial\tau^\alpha \tag{39}$$

where $\mathcal{H}(\theta, \tau^\alpha, p_\alpha)$, given by (20), is the *T-Hamiltonian* associated with the given *T-Lagrangian.*

Remark. This construction, namely, establishing the family of *temporally equidistant* hypersurfaces together with the congruence of curves belonging to it, is tantamount to constructing the *complete figure of the problem* in the calculus of variations.

IV. The duration function on T-Space. Let us suppose that we are given a family of hypersurfaces $S(\theta, \tau^\alpha) = \chi$ on T-Space which are solutions to (38) with (39) and are thus *temporally equidistant.* This suggests, according to remark 2 above, that we establish a firm relationship between $\mathrm{dur}(P, Q)$ on the Set of Instants $\mathcal{I}$ and $I = \int_p^q \mathcal{L}(\theta, \tau^\alpha, \tau'^\alpha)d\theta = \chi_2 - \chi_1$ on the T-Space. To this end we are motivated by the desire to maintain a clear connection between a *time track* on $\mathcal{I}$ and curves in T-Space, for the *duration function*, as introduced by DeVito, adheres to properties common to "observable time." By constructing the *complete figure of the problem* we are guided by the geometric simplicity of *temporally equidistant hypersurfaces.* That is, an "observable time" difference (ΔT) should not only be equivalent to all observers on a *time track* on $\mathcal{I}$, but should also be equivalent when considered with respect to the *T-Space* of that *time track.* In this regard let us now require that the real number $|\chi_2 - \chi_1|$ in (38) be identified with the real number defined by $\mathrm{dur}(P, Q)$ in (1) for $P, Q \in \mathcal{C}(\mathcal{I})$. We refer to this identification as the *local temporal equivalent condition.* Thus let fixed points $P, Q \in \mathcal{C}(\mathcal{I})$ with $\phi(P) = (\tau_1^\alpha)$ and $\phi(Q) = (\tau_2^\alpha)$ be identified with the endpoints $\mathbf{p}$,$\mathbf{q}$ of γ a member of the T-congruence with respective coordinates $(\theta_1, \tau_1^\alpha)$ and $(\theta_2, \tau_2^\alpha)$ which determine the members χ_1 and χ_2 of the family of hypersurfaces through which γ passes and by the *local temporal equivalence condition* write,

$$\mathrm{dur}(P, Q) = |\rho| = |\chi_2 - \chi_1| \tag{40}$$

which immediately satisfy (1a,b) for $P, Q \in \mathcal{C}(\mathcal{I})$.

Lemma 3. The *local temporal equivalence condition* constitutes a duration function restricted to $\mathcal{C}(\mathcal{I})$.

APPENDIX

1. The proof of the theorem follows simply by partial differentiation of (6) with respect to y_j followed by substitution of (5), noting that $\xi^j = \xi^j(y_k)$. By virtue of (3):

$$\partial h/\partial y_j = \xi^k \delta^j_k + y_k \partial \xi^k/\partial y_j - (\partial f/\partial \xi^k)(\partial \xi^k/\partial y_j). \ \square$$

Also, since T followed by T^{-1} is the identity I we have

$$\partial(y_{,1}, \ldots, y_{,n})/\partial(\xi_{,1}, \ldots, \xi_{,n}) \cdot \partial(\xi_{,1}, \ldots, \xi_{,n})/\partial(y_{,1}, \ldots, y_{,n}) = 1,$$

which, by (5) and (7), yields (8).

2. The corollary follows immediately with (9) written in terms of λ as $f(\xi^j, \lambda) + h(y^j, \lambda) = \xi^k y_k$ regarding (ξ^j, λ) as independent variables then $\partial f/\partial \lambda + \partial h/\partial \lambda + (\partial h/\partial y^j)(\partial y^j/\partial \lambda) = \xi^k(\partial y_k/\partial \lambda)$ so that from (7) the result now follows.

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